Problem 2.21

A gun can fire shells in any direction with the same speed v_o. Ignoring air resistance and using cylindrical polar coordinates with the gun at the origin and z measured vertically up, show that the gun can hit any object inside the surface

$$z = \frac{v_o^2}{2g} - \frac{g}{2v_o^2}\rho^2.$$

Describe this surface and comment on its dimensions.

Solution

Newton's second law gives the equation of motion for the projectile. Expand the equation in cylindrical coordinates (ρ, ϕ, z).

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \begin{cases} \sum F_\rho = m\left[\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2\right] \\ \sum F_\phi = m\left(\rho\frac{d^2\phi}{dt^2} + 2\frac{d\rho}{dt}\frac{d\phi}{dt}\right) \\ \sum F_z = m\frac{d^2z}{dt^2} \end{cases}$$

Since there's no air resistance, the only force to consider is the one due to gravity.

$$\begin{cases} 0 = m\left[\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2\right] \\ 0 = m\left(\rho\frac{d^2\phi}{dt^2} + 2\frac{d\rho}{dt}\frac{d\phi}{dt}\right) \\ -mg = m\frac{d^2z}{dt^2} \end{cases}$$

There's no rotational motion about the z-axis, so all the derivatives of ϕ are zero.

$$\begin{cases} 0 = m\frac{d^2\rho}{dt^2} \\ 0 = 0 \\ -mg = m\frac{d^2z}{dt^2} \end{cases}$$

Divide both sides of each equation by m.

$$\begin{cases} \dfrac{d^2\rho}{dt^2} = 0 \\ \\ \dfrac{d^2z}{dt^2} = -g \end{cases}$$

Integrate both sides with respect to time. Suppose the projectile is launched with speed v_o at an angle θ above the xy-plane. Then the velocity in the radial direction is $v_o\cos\theta$, and the velocity in the vertical direction is $v_o\sin\theta$.

$$\begin{cases} \dfrac{d\rho}{dt} = v_o\cos\theta \\ \\ \dfrac{dz}{dt} = -gt + v_o\sin\theta \end{cases}$$

Integrate both sides with respect to time again.

$$\begin{cases} \rho(t) = (v_o\cos\theta)t + \rho_o \\ \\ z(t) = -\dfrac{1}{2}gt^2 + (v_o\sin\theta)t + z_o \end{cases}$$

The projectile is launched from the origin, so $\rho_o = 0$ and $z_o = 0$.

$$\begin{cases} \rho(t) = (v_o\cos\theta)t \\ \\ z(t) = -\dfrac{1}{2}gt^2 + (v_o\sin\theta)t \end{cases}$$

Since we want the equation of a surface, eliminate t: Solve the first equation for t,

$$t = \frac{\rho}{v_o\cos\theta},$$

and plug it into the second equation.

$$z = -\frac{1}{2}g\left(\frac{\rho}{v_o\cos\theta}\right)^2 + (v_o\sin\theta)\left(\frac{\rho}{v_o\cos\theta}\right)$$

$$z = -\frac{g\rho^2}{2v_o^2}\sec^2\theta + \rho\tan\theta$$

In order to find the angle that maximizes the projectile's height, take the derivative of z with respect to θ,

$$\frac{\partial z}{\partial\theta} = -\frac{g\rho^2}{2v_o^2}(2\sec^2\theta\tan\theta) + \rho\sec^2\theta$$

$$= \rho\sec^2\theta\left(1 - \frac{g\rho}{v_o^2}\tan\theta\right),$$

set it equal to zero,

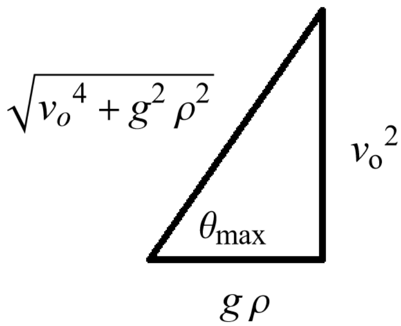
$$\rho \sec^2 \theta_{\text{max}} \left(1 - \frac{g\rho}{v_o^2} \tan \theta_{\text{max}}\right) = 0,$$

and solve for θ_{max}.

$$1 - \frac{g\rho}{v_o^2} \tan \theta_{\text{max}} = 0$$

$$\tan \theta_{\text{max}} = \frac{v_o^2}{g\rho}$$

Draw the implied right triangle.



Finally, plug θ_{max} into the formula for the projectile's height to get the equation of the surface that marks the furthest the projectile can reach.

$$\begin{aligned}
z &= -\frac{g\rho^2}{2v_o^2} \sec^2 \theta_{\text{max}} + \rho \tan \theta_{\text{max}} \\
&= -\frac{g\rho^2}{2v_o^2} \left(\frac{\sqrt{v_o^4 + g^2\rho^2}}{g\rho}\right)^2 + \rho \left(\frac{v_o^2}{g\rho}\right) \\
&= -\frac{g\rho^2}{2v_o^2} \left(\frac{v_o^4 + g^2\rho^2}{g^2\rho^2}\right) + \frac{v_o^2}{g} \\
&= -\frac{g\rho^2}{2v_o^2} \left(\frac{v_o^4}{g^2\rho^2} + 1\right) + \frac{v_o^2}{g} \\
&= -\frac{v_o^2}{2g} - \frac{g\rho^2}{2v_o^2} + \frac{v_o^2}{g} \\
&= \frac{v_o^2}{2g} - \frac{g}{2v_o^2}\rho^2
\end{aligned}$$

This surface is a paraboloid. In the case where the projectile is shot straight up along the z-axis ($\rho = 0$), it achieves a maximum height of $v_o^2/(2g)$. The distance from the origin where the maximum height is zero is the range.

$$0 = \frac{v_o^2}{2g} - \frac{g}{2v_o^2}R^2 \quad \rightarrow \quad R^2 = \frac{v_o^4}{g^2} \quad \rightarrow \quad R = \frac{v_o^2}{g}$$